

“Lecture” Note
Engg. Mathematics-2
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Integration

Studying the inverse process of differentiation is called integration - when a funⁿ is given this process tries to determine another funⁿ whose derivative is the given funⁿ. The branch of calculus which studied about integration its & application is called integral calculus.

Antiderivative $\Rightarrow$

If $g(x)$ is the derivative of $f(x)$ then $f(x)$ is said to be an antiderivative or integral of $g(x)$.

Ex
 $\cos x$ is the derivative of $\sin x$, $\sin x$ is the derivative of $\cos x$. The symbol $\int$ (an elongated s) is used to denote the operation of integration & called the integral sign symbolically, written as

$$\int \cos x \, dx = \sin x$$

The funⁿ $(\cos x)$ to be integrated dx denotes the fact that the integration is to be performed with respect to x (i.e. x is the variable of integration).

observe that each one of $\sin x$, $\cos x$, $\sin x \cdot 5$, $\cos x$ has the same antiderivative so antiderivative of a function is unique.

Antiderivative of $\cos x$ is the $\sin x + k$, k is any constant if $g(x)$ is the derivative of $f(x)$ ($f(x) + k$) then $f(x) + k$ denotes the family of all antiderivative constant. therefore $f(x) + k$ is called the indefinite integral of $g(x)$. we write $\int g(x) dx$ where k is any constant = $f(x) + k$ of integration which is arbitrary we also use A, B, C, a, b, c etc. for constant of integration

Formula

$$\int x^n dx = \frac{x^{n+1}}{n+1} + k \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln |x| + k$$

$$\int \cos x \cdot dx = \sin x + k$$

$$\int \sin x \cdot dx = -\cos x + k$$

$$\int \sec^2 x dx = \tan x + k$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + k$$

$$\int \sec x \cdot \tan x dx = \sec x + k$$

$$\int \operatorname{cosec} x \cdot \cot x dx = -\operatorname{cosec} x + k$$

$$\int e^x dx = e^x + k$$

$$\int a^x dx = \frac{a^x}{\ln a} + k$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{cosech}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech} x \cdot \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{cosech} x \cdot \coth x \, dx = -\operatorname{cosech} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \quad \text{or} \quad -\cos^{-1} x + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C \quad \text{or} \quad \cot^{-1} x + C$$

$$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C \quad \text{or} \quad \operatorname{cosec}^{-1} x + C$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C$$

ALGEBRA OF INTEGRALS

$$(i) \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$$(ii) \int \lambda f(x) \, dx = \lambda \int f(x) \, dx, \text{ for a constant } \lambda.$$

$$\int \tan x \cdot dx = -\log \cos x + C$$

$$\int \cot x \cdot dx = \log \sin x + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \operatorname{cosec} x \, dx = \ln |\sec x - \cot x| + C$$

using standard method

• using substitution

• decomposition in to sum

• definite integral

• integration by part

find the area by using integration.

m integration by using trigonometric form

$$1. \sin mx \cdot \cos nx = \frac{1}{2} [2 \sin mx \cos nx] \\ = \frac{1}{2} [\sin(m+n)x + \sin(m-n)x]$$

$$2. \sin mx \cdot \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$$

$$3. \cos mx \cdot \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

$$4. \sin mx \cdot \cos nx \cdot \cos kx$$

$$= \frac{1}{4} [\sin(m+n+k)x + \sin(m-n+k)x +$$

$$\sin(m+n-k)x + \sin(m-n-k)x]$$

~~work~~

EXERCISE 13 (C)

$$1. (i) \int \sin 4x \cos 3x \, dx$$

$$= \int \frac{1}{2} [2 \sin 4x \cos 3x] \, dx$$

$$= \int \frac{1}{2} [\sin 7x + \sin x] \, dx$$

$$= \frac{1}{2} \int \sin 7x \, dx + \int \sin x \, dx$$

$$= \frac{1}{2} \left[-\frac{\cos 7x}{7} \right] + \left[-\frac{\cos x}{2} \right] + C$$

$$= -\frac{1}{14} \cos 7x - \frac{1}{2} \cos x + C$$

$$(ii) \int \cos 3x \cos 2x \, dx$$

$$\frac{1}{2} \int [\cos 3x + \cos 5x] \, dx$$

$$\frac{1}{2} \int \cos 3x \, dx + \int \cos 5x \, dx$$

$$\frac{1}{2} \cdot \frac{\sin 8x}{8} + \frac{1}{2} \cdot \frac{\sin 7x}{7} + C$$

$$\frac{1}{6} \sin 3x + \frac{1}{14} \sin 7x + C$$

$$\text{iii) } \int \sin x \cos 4x \, dx$$

$$\frac{1}{2} \int \sin 5x \, dx + \int \sin (-3x) \, dx$$

$$\frac{1}{2} \cdot \frac{-\cos 5x}{5} + \frac{1}{2} \cdot \frac{+\cos 3x}{+3} + C$$

$$\frac{1}{6} \cos 3x - \frac{1}{10} \cos 5x + C$$

$$\text{iv) } \int \sin 6x \sin 8x \, dx$$

$$\frac{1}{2} \int \cos 2x \, dx - \frac{1}{2} \int \cos 14x \, dx$$

$$\frac{1}{2} \cdot \frac{\sin 2x}{2} - \frac{1}{2} \cdot \frac{\sin 14x}{14} + C$$

$$\frac{1}{6} \sin 2x - \frac{1}{18} \sin 14x + C$$

$$\text{v) } \int \cos 4x \cos 5x \sin 2x \, dx$$

$$\int \sin 2x \cos 5x \cos 4x \, dx$$

$$= \frac{1}{4} \left(\int \sin 11x \, dx + \int \sin (-7x) \, dx + \int \sin 3x \, dx + \int \sin x \, dx \right)$$

$$= \frac{1}{4} \left(\frac{\cos 11x}{11} - \frac{\cos 7x}{7} + \frac{\cos 3x}{3} + \cos x \right)$$

$$= \frac{\cos 11x}{44} - \frac{\cos 7x}{28} + \frac{\cos 3x}{12} + \frac{\cos x}{4}$$

When the identities consist of trigonometric identities which can easily be interchanged. Some identities useful for this purpose are

To apply the above formula in integration, we shall require the formula. we proved in trigonometry. let us recall them:

$$1) \sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$$ii) \sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$iii) \cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$$

$$iv) \cos A \cdot \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

$$v) \sin A \cdot \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$vi) \sqrt{1 + \sin A} = \cos \frac{A}{2} + \sin \frac{A}{2}$$

$$vii) \sqrt{1 - \sin A} = \cos \frac{A}{2} - \sin \frac{A}{2}$$

$$viii) \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$ix) \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$x) \sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$$

$$xi) \cos^3 \theta = \frac{3}{4} \cos \theta + \frac{1}{4} \cos 3\theta$$

can be established that using ~~the~~ Trigonometric Function

$$\sin^{-1} \frac{x}{a} = \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1} \right|$$

$$\cos^{-1} \frac{x}{a} = \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right|$$

The result obtained above can be used on standard formula

$$1. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

$$2. \int \frac{1}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$3. \int \frac{1}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C$$

$$4. \int \frac{1}{\sqrt{x^2 + a^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right| + A$$

$$= \sinh^{-1} \frac{x}{a} + B$$

$$5. \int \frac{1}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + A$$

$$= \cosh^{-1} \frac{x}{a} + B$$

Definite Integrals

upper limit
 $\int_a^b f(x) dx = f(x) \Big|_a^b$
lower limit

$$= f(b) - f(a)$$

Properties of Definite Integrals

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \int_a^b f(x) dx$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx \quad \text{order of integration}$$

$$3) \int_a^a f(x) dx = 0$$

$$4) \int_a^b k f(x) dx = k \int_a^b f(x) dx$$

$$5) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$a < c < b$

$$6) \int_a^b f(x) dx = \int_0^a f(a-x) dx$$

in case of definite, the result

$$\int_a^b [g(x) + h(x)] dx = \int_a^b g(x) dx + \int_a^b h(x) dx$$

Integration by parts

If u and v are two differentiable funⁿ of x then

$$\frac{d}{dx}(u.v) = v \cdot \frac{d}{dx}u + u \cdot \frac{d}{dx}v$$

$$v \cdot \frac{du}{dx} = D(uv) - u \cdot \frac{dv}{dx}$$

taking in both integration both side.

$$\int v \cdot \frac{du}{dx} = \int \frac{d}{dx}(u.v) - \int u \cdot \frac{dv}{dx}$$

Let, this result can be written as

$$\frac{du}{dx} = w$$

$$w = \int u \cdot dx$$

this rule is called rule of integration of by parts.

$$\int v \cdot w dx = \int (w dx) v - \int \int w dx \frac{dv}{dx} dx$$

this rule is called rule of integration by parts and is used to integrate product of 2 funⁿs.

integral product of two funⁿ = 1st funⁿ x

integral of 2nd funⁿ - integral of

(difference of 1st funⁿ) x integral of 2nd funⁿ

is difference open crossing of 1st and 2nd funⁿs.

usually 1st & 2nd funⁿ is made by table. ETAL

is ~~trans for~~ $E = \text{Exponation fun}^n = e^x$

$T = \text{trigonometri fun}^n = \sin x, \cos$

$A = \text{Algebraic fun}^n = x^2, x^3, x^4$

$L = \text{Logarithm fun}^n = \log(x), \log(y)$

$I = \text{inverse trigonometri fun}^n = \sin^{-1}x, \cos^{-1}x$

But, it applied according to

Example-1

$$\int x \cos x dx$$

$$\text{ans} \rightarrow x \int \cos x dx - \int (x \cdot dx) \cos x dx$$

$$x \sin x - \int 1 \cdot \sin x dx$$

$$x \sin x - (-\cos x) + C$$

$$x \sin x + \cos x + C$$

Example-2

$$\int x \sin x dx$$

$$\text{ans} \rightarrow x \int \sin x dx - \int (x \cdot dx) \sin x$$

$$x(-\cos x) - \int (1 \cdot -\cos x) dx$$

$$-x \cos x - \int -\cos x dx$$

$$-x \cos x + \sin x + C$$

Example-3

$$\int x \tan^{-1} x dx$$

$$\int \tan^{-1} x \cdot 1 dx$$

$$\tan^{-1} x \int 1 dx - \int \frac{d}{dx} \tan^{-1} x \int 1 dx$$

$$\tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x dx$$

$$x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$\text{Let } 1+x^2 = z$$

$$2x \cdot dx = dz$$

$$x dz = \frac{1}{2} dz$$

$$x \tan^{-1} x - \int \frac{1}{z} \cdot \frac{1}{2} dz$$

$$x \tan^{-1} x - \frac{1}{2} \int \frac{1}{z} dz$$

$$\frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{1}{1+x^2} dx$$

Exercise

10. $\int x^2 \cos x dx$

$$x^2 \int \cos x dx = \int \left(\frac{d}{dx} x^2 \right) (\cos x) dx$$

$$x^2 \sin x = \int 2x \cdot \sin x dx$$

$$x^2 \sin x = 2 \int x \cdot \sin x dx$$

$$x^2 \sin x = 2 \int x \cdot \sin x dx$$

when 1, $x^2 \sin x = 2 (-x \cos x + \sin x)$

$$I = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

$$I_1 = x \int \sin x dx - \int \frac{d}{dx} (x) (\sin x) dx$$

$$I_1 = -x \cos x + \int \cos x dx$$

$$I_1 = -x \cos x + \sin x$$

b) $\int x \tan^{-1} x dx$

$$\tan^{-1} x \int x dx = \int \left(\frac{d}{dx} \tan^{-1} x \right) \left(\frac{x^2}{2} \right) dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \int \frac{x^2 - 1 + 1}{1+x^2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \left[\int \frac{1+x^2}{1+x^2} dx - \int \frac{1}{1+x^2} dx \right]$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \left[\int dx - \tan^{-1} x \right]$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

$$\int (x^2) (e^x dx) dx$$

$$e^x dx$$

$$e^x dx$$

$$e^x dx$$

$$e^x dx$$

$$\int (x^2) (e^x dx) dx = \int \left(\frac{d}{dx} (x^2) \right) (e^x) dx$$

$$x^2 \cdot \frac{1}{2} = \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \int \frac{x^2 + 1 - 1}{1+x^2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \frac{1}{2} \int \frac{1+x^2}{1+x^2} dx - \int \frac{1}{1+x^2} dx$$

$$\frac{x^2 \tan^{-1} x}{2} = \left[\frac{1}{2} \int dx - \int \frac{1}{1+x^2} dx \right]$$

$$\frac{x^2 \tan^{-1} x}{2} = \left[\frac{x}{2} - \int \frac{1}{1+x^2} dx \right]$$

$$\int_a^b \lambda g(x) dx = \lambda \int_a^b g(x) dx$$

$$\textcircled{7} \int_a^b f(x) dx = \int_a^0 f(x) dx + \int_0^b f(x) dx, f(-x) = f(x), f(-x) = -f(x)$$

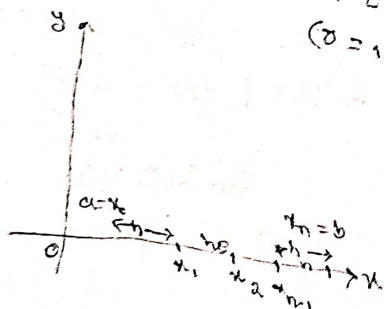
$$\textcircled{8} \int_a^{2a} f(x) dx = \int_a^0 f(x) dx + \int_0^{2a} f(x) dx, f(2a-x) = f(x)$$

$$f(2a-x) = -f(x)$$

$$\textcircled{9} \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

Integration can be considered as a process of summation. A funⁿ $f(x)$ continuous in the interval $[a, b]$. Let the interval be divided into n subintervals of lengths $h_1, h_2, \dots, h_n$ by the pts $x_1, \dots, x_{n-1}$ s.t $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ then $h_r = x_r - x_{r-1}$ ($r = 1, 2, \dots, n$)

Let x_0 be any point in $[x_{r-1}, x_r]$ ($r = 1, \dots, n$)



defining $\lambda = \inf f(x)$...
 if the limit of λ exists as $n \rightarrow \infty$ then the maximum of λ is called the definite integral of $f(x)$ from a to b it is denoted by $\int_a^b f(x) dx$

$$\text{so } \int_a^b f(x) dx = \lambda + C \text{ where } C = f(x)$$

Fundamental Theorem of Integral Calculus
 This establishes a relation between definite and indefinite integrals.

Statement
 If $f(x)$ is continuous in the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$ then

$$\int_a^b f(x) dx = F(b) - F(a)$$

From indefinite integration, it is known that $\int f(x) dx = F(x) + C$ where C is an arbitrary constant. For any C , the difference of the values of $F(x) + C$ at $x=b$ and at $x=a$ is

$$[F(x) + C]_{x=b} - [F(x) + C]_{x=a} = F(b) + C - F(a) - C = F(b) - F(a)$$

$$\frac{x^2 - 9}{x - 3}$$

$$\frac{x^2 + 2x}{x - 2}$$

$$\frac{(x+3)(x-3)}{(x-3)}$$

$$8 + 3 = x$$

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$$

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)}$$

$$1 + 1 + 1 = 3$$

C-3

$$\lim_{x \rightarrow 2} \frac{x-2}{x^4 - 16}$$

$$\lim_{x \rightarrow 2} \frac{x-2}{x^4 - 2^4}$$

Even

" trigonometric "

$$\lim_{x \rightarrow 0} \frac{1}{x} \frac{\sin^{-1} 2x}{1+x^2}$$

$$\lim_{x \rightarrow 1} \frac{1}{x} \frac{2 \tan^{-1} x}{1+x^2}$$

$$\therefore \frac{\sin^{-1} 2x}{1+x^2}$$

$$2. \lim_{x \rightarrow 1} \frac{2 \tan^{-1} x}{x}$$

$$2 \tan^{-1} x$$

2

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x}$$

$$\lim_{x \rightarrow 0} \frac{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}{\sin^{-1} x \left[(\sqrt{1+x}) + (\sqrt{1-x}) \right]}$$

$$\lim_{x \rightarrow 0} \frac{(1+x) - (1-x)}{\sin^{-1} x \left[\sqrt{1+x} + \sqrt{1-x} \right]}$$

$$\lim_{x \rightarrow 0} \frac{2x}{\sin^{-1} x \left[\sqrt{1+x} + \sqrt{1-x} \right]}$$

$$2. \lim_{x \rightarrow 0} \frac{x}{\sin^{-1} x} \times \frac{1}{\sqrt{1+x} + \sqrt{1-x}}$$

$$2. \lim_{x \rightarrow 0} \frac{1}{\frac{\sin^{-1} x}{x}} \times \frac{1}{(\sqrt{1+x} + \sqrt{1-x})}$$

$$\lim_{x \rightarrow 0} \frac{10^x - 2^x - 5^x + 1}{x^2} \quad \text{form } \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{3^x - 2^x}{x} \quad \text{form } \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x} \quad \text{form } \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{10^x - 2^x - 5^x + 1}{x^2} \quad \text{form } \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{3^x + 3^{-x} - 2}{x^2} \quad \text{form } \frac{0}{0}$$

$$\textcircled{6} \quad \lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2} \quad \text{form } \frac{0}{0}$$

$$\textcircled{7} \quad \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x} \quad \text{form } \frac{0}{0}$$